EQUIVALENCE OF SYLLOGISMS

• “two syllogisms are equivalent if and only if they have the same models”

**Plan:** to give two definitions of when syllogisms are equivalent (one syntactic, one semantic), using group-theoretic methods.

**GROUPS**
- G is a nonempty set of permutations
- · is a binary operation in G (sequential application)
- (G, · ) is called a group if the following three axioms hold:

  **Associativity:** a(bc) = (ab)c for all a,b,c ∈ G.

  **Identity:** There exists an e ∈ G such that ea = a for all a ∈ G.
  (the identity element e can also be denoted by 1, since the binary operation in G is written multiplicatively)

  **Inverses:** For every a ∈ G there exists an a¹ ∈ G such that a¹a = e.

**Example of a group: Symmetries of an equilateral triangle.**

Let X be the set of points on the perimeter of an equilateral triangle
- Then G is a group acting on X.
- A permutation (or transformation) is called a symmetry when it preserves distances (i.e., it preserves the triangle)?

Three of the symmetries move the vertices in the following manner (counterclockwise rotations):

- **e**
  - 1 → 1
  - 2 → 2
  - 3 → 3
  - Also written as (11)(22)(33)

- **a**
  - 1 → 2
  - 2 → 3
  - 3 → 1
  - Also written as (12)(23)(31)

- **b**
  - 1 → 1
  - 2 → 2
  - 3 → 3
  - a²b
  - 2 → 2
  - 3 → 3
  - ab
  - 3 → 3
  - 1 → 2
  - 2 → 1

**Reflections**
- The six symmetries of an equilateral triangle: e, a, a², b, a²b, ab
- G: e, a, b
- G forms a symmetry group, since all the permutations maintain symmetry.
- Any symmetry of the equilateral triangle is determined by its effect on three vertices. Therefore, the set of six symmetries is a complete list of symmetries of an equilateral triangle.

**Orbits**
Let G be a group acting on a set X, and let x ∈ X. Then the set
Gx = {ax | a ∈ G}

is called the orbit of x in G.
- Orbit(a): a, a², e
  Namely, with a and · those are the symmetries one can get (a, (a·a), (a·a·a))
- Orbit(b): b, e
- Orbit(e): e
- To get all six symmetries both permutation a and permutation b are needed (and nothing more):
  Orbit(a,b): e, a, a², b, a²b, ab.
- a and b are the generators of G (for these two permutations are sufficient to generate the group).


Starting point: correct application of classical conversion and RAA to a syllogism will result in an equivalent syllogism.
Problem: “correct application” is restricted. Also, transformations need to be one-to-one, because every transformation needs to have an inverse.

1. **Categorical syllogisms**

   - An inference of the form \((p \land q \rightarrow r)\), where \(p\), \(q\) and \(r\) are categorical propositions (or statement), i.e., statements containing two categorical terms (they designate classes) in which the latter (the predicate term) is affirmed or denied of the former (the subject term).

   - Categorical statements affirm some relations between their terms.
     Thus you get four statement types: A, E, O, I

   - Read Asp (or SaP) as: All S are P (as opposed to S belongs to all P)

**The traditional figures.** There are 4 ways for the statements to share terms:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>major</td>
<td>MP</td>
<td>PM</td>
<td>MP</td>
<td>PM</td>
</tr>
<tr>
<td>minor</td>
<td>SM</td>
<td>SM</td>
<td>MS</td>
<td>MS</td>
</tr>
<tr>
<td>SP</td>
<td>SP</td>
<td>SP</td>
<td>SP</td>
<td></td>
</tr>
</tbody>
</table>

- 4 (figures) \(\cdot (4 \cdot 4 \cdot 4)\) (statement types) = 256 forms.

**How is Richman going to represent these?**
Triangles represent the figures, i.e., they represent the order of premises and the order of the terms. The vertices correspond to the terms; the sides to the premises. To get a syllogism from a form we assign a class to each vertex and we assign a statement type to each side – the base representing the conclusion.
Syntax?
- Equilateral triangles
- Set of constant symbols: S, P, M (denoting the terms) – are assigned to the vertices of the triangle.
- Statement type symbols: A, E, O, I, a, e, o, i – are assigned to the sides of the triangle.
- \( \equiv \) - assigned to the insides of the of the triangle
- Figure = triangle with an arrow on each side of the triangle
- Form = figure with a type assigned to each side of the triangle
- Instantiated form = form with a constant symbol assigned to each vertex of the triangle
- Permutation symbols: \( Oi, Ci, Ti \).
- If \( \alpha \) is a form, then \( Oi(\alpha) \) is a form.

2. Transformations and nonstandard syllogisms

The classical transformations:
- Conversion (to interchange premises):
  1. Subject and predicate of an E or I statement are interchanged.
  2. If needed, the two premises are interchanged.

In terms of triangles: reverse the arrow on a side (conversion), and reflect through the altitude of one of the vertices if needed (exchange of premises).

Applying conversion to the conclusion of EAE-1 (Celarent):

\[
\begin{align*}
EAE-1 & \\
MeP & \quad MeP & \quad SaM \\
SaM & \quad SaM & \quad MeP \\
SeP & \quad e-conversion of PeS & \quad exchanging premises PeS
\end{align*}
\]

- RAA:
  1. One of the premises and the conclusion are interchanged and negated.
  2. If needed, the two premises are interchanged.

In terms of triangles: flip the triangle around one of the base vertices, and change the letters on the adjacent sides, so as to negate - and exchange premises if needed.

With conversion and RAA the 15 "valid" forms can be obtained from Barbara and Celarent. However, Richman has two objections to the classical transformations:
1. The interchange of premises seems a little hokey. [p.5]

2. The transformations are not one-to-one, hence do not form a group. [p. 5]
   (thus: RAA and conversion do not form a group?)

For instance: AAA-1 (Barbara) → OAO-3 (Bocardo)
               AOO-2 (Baroco)  RAA

**Introducing additional figures.**
Allowing the major premise to appear in the second position. This will necessitate four *nonstandard* figures:

<table>
<thead>
<tr>
<th>1'</th>
<th>2'</th>
<th>3'</th>
<th>4'</th>
</tr>
</thead>
<tbody>
<tr>
<td>SM</td>
<td>SM</td>
<td>MS</td>
<td>MS</td>
</tr>
<tr>
<td>MP</td>
<td>PM</td>
<td>MP</td>
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<td>SP</td>
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</table>

**Nonstandard figures**: flip the triangle around its vertical axis of symmetry so the premises are interchanged.

- **RAA**: “The nonstandard figures seem to provide the right setting for RAA.” (6)

<table>
<thead>
<tr>
<th>AOO-2</th>
<th>AOO-3'</th>
<th>OAO-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PaM</td>
<td>→</td>
<td>SaP</td>
</tr>
<tr>
<td>SoM</td>
<td>RAA</td>
<td>SoM</td>
</tr>
<tr>
<td>SoP</td>
<td></td>
<td>PoM</td>
</tr>
</tbody>
</table>

  AAA-1 1:1 OAO-3
  MaP    SoP
  SaM    SaM
  SaP    MoP

Allowing nonstandard figures will make RAA a one-to-one transformation!

**Introducing additional statement types** (De Morgan types).

- **Conversion**
  We can only convert statements of type E and I. Therefore, there is -as yet-no transformation on the space that corresponds to a conversion of a statement that is not of type E or I.
  How to deal with conversion of A- and O-statements? Answer: introduce additional statement types (De Morgan):
  - In case of A and O statements: interchange subject and predicate and change the type of the statement to a or o.
• These types are said to “quantify the predicate”.
  \[ a: \text{SaP} = \text{PaS} \]
  \[ o: \text{SoP} = \text{PoS} \]

• In a and o statements, what kind of relation is being affirmed between the terms?
  Changing the case of A, O, a or o commutes with negating (\(\neg\)).

Conversion of the conclusion in terms of **triangles**: reverse the arrow on the base and interchange the letters A and a, and the letters O and o.

• **Obversion**

\(\neg P\) denotes the complement of the class denoted by P:

\[ \text{SaP} = \text{Se}\neg P \quad (\text{Every man is animate} = \text{No man in inanimate}) \]
\[ \text{SiP} = \text{So}\neg P \quad (\text{Some men are strong} = \text{Some men are weak}) \]

**Problem**: obversion cannot be applied to syllogisms of the fourth figure, since we can only obvert at those terms that are double predicates (i.e., terms that appear only as predicate)

Therefore, Richman is going to use De Morgan's four statement types: a, o, i, e.

  \[ e: \text{everything is neither a cat nor a dog} \]
  \[ i: \text{there is something that is neither a cat nor a dog} \]

3. **The syllogism group**

[p. 8]

Group Generators: \(O_i\) (obversion), \(C_i\) (conversion), \(T_i\)

Identity transformation: 1

**Result**: When are two syllogism equivalent? When they can be transformed into each other by an element of G (i.e., sequential application of \(O_i\), \(C_i\), and \(T_i\))!

5. **Orbits of the syllogism group G**

• Let \(X\) be the set of (4096) syllogistic forms; G, the group generated by obversions, conversions and RAA’s, acting on \(X\).

• If \(\alpha\) is a syllogism (form) then,

\[ \text{Orbit}(\alpha) : \{g\alpha \mid g \in G\} \]

Two syllogisms are in the same orbit of G if some element of G transforms the one into the other.
• Orbit(Barbara): all syllogisms you get by (sequential) application of \(Ci, Oi\) and \(Ti\).

Richman wants to show two things:

1. If two standard syllogisms are in the same orbit of \(G\), then we can transform one into the other by sequential application of obversion, conversion and RAA without leaving the subspace of standard syllogisms (i.e., the space of 512 forms – that include nonstandard figures).

2. If two syllogisms are equivalent, then they are in the same orbit of \(G\).

- Richman shows that every orbit of \(G\) contains a standard syllogism (p. 13).
- A standard form \(\alpha\) cannot be equivalent to its full negation.
- Richman verifies that elements of different orbits are inequivalent – computer assisted (pp. 14-6)

6. Orbits of subgroups of \(G\)

Ascertaining the structure of \(G\) as an abstract group

- By restricting the allowable transformations, we get different notions of equivalence of syllogisms
- The equivalence classes are the orbits of various subgroups of \(G\).
- In this section Richman studies those orbits, and the corresponding orbits in the space of 256 standard syllogisms

- \(Barbara\) is in an orbit of size 48, containing \(Barbara\), \(Baroco\) en \(Bocardo\).

- \(Celarent\) is in an orbit of size 48, containing the remaining twelve valid standard syllogisms.

7. Antilogisms

Point: a more efficient way to analyze equivalence of syllogisms.

- Antilogisms say: \(\neg (p \& q \& r)\)
- When is an antilogism valid?
  - Each term is distributed in some statements
  - There are two positive statements and one negative statement
  - There are two universal statements and one particular statement (or just not three universal statements).

Should we be satisfied with Richman’s model?

- Equivalent syllogisms can be transformed into each other by an element of \(G\).
- Group G is generated by obversions, conversions and indirect reductions.
- The orbits of Barbara and Celarent contain all valid syllogisms
- He has not assumed existential import
- He had to allow additional figures
- He had to allow additional statement types

**Aristotle, Prior Analytics**

"I call that a **term** into which the premise is resolved, i.e., both the predicate and that of which it is predicated..."

"Every premise states that something either is or must be or may be the attribute of something else..."

"A **syllogism** is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so."

"It is evident also that in all the figures, whenever a proper syllogism (i.e., of the first figure) does not result, if both the terms are affirmative or negative nothing necessary follows at all, but if one is affirmative, the other negative, and if the negative is stated universally, a syllogism always results relating the minor to the major term..."

**Conversion**

"It is necessary then that in universal attribution the terms of the negative premiss should be convertible, [...] the terms of the affirmative must be convertible, not however, universally, but in part..."

"But if every B is A then some A is B. For if no A were B, then no B could be A. But we assumed that every B is A." (**existential import**)"

"... but the particular negative need not **convert**, for if some animal is not man, it does not follow that some man is not animal."

**The Tree of Porphyry**
Thus *animate* and *inanimate* are differentiae of the genus *body*. Each definition takes the form:

- **Definiendum** $=$$_{df}$ differentia + genus