Graphs, distances and eigenvalues

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1 Introduction

We have developed a tool with which we can derive bounds on subsets of vertices in a graph satisfying some distance property, in terms of the (Laplace) eigenvalues of the graph. The significance of such a bound is obvious when the related problem is NP-complete, and the determination of a good solution will be computationally hard, for instance, when we bound the sizes of two sets of vertices at some given distance, or in particular, when we bound the size of two equally large sets of vertices with no edges in between. When we are dealing with a relatively easy problem, like the determination of the diameter of a graph, a bound in terms of (some of) the eigenvalues of the graph can still be useful. Of course, when given a graph, we should not compute its eigenvalues, and then derive the diameter bound, when it is much easier to find the diameter explicitely with a polynomial-time algorithm. However, sometimes we do not know the full structure of a graph, while we may have some information about its eigenvalues. This is for example the case with so-called Ramanujan graphs, graphs which are known to have good expanding properties (cf. [9]), and which therefore can be used to build good (and large) information networks (cf. [1]).

In this paper we consider undirected graphs. The Laplace eigenvalues of such a graph are the eigenvalues of the associated Laplace matrix Q, which is a square matrix with rows and columns labelled by the vertices of the graph, defined by $Q_{xx} = d_x$, and $Q_{xy} = -A_{xy}$ for $x \neq y$, where d_x is the vertex degree of x, and A_{xy} denotes the number of edges between x and y. The Laplace matrix is a positive semidefinite matrix.

The results in this paper are mainly from [3] and [4] (see also Chapter 5 of the author's thesis [2]).

2 The tool

Let P_m be the set of polynomials p with real coefficients of degree m such that $p(0) = 1$. Our main tool will be the following theorem.

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Theorem 1 *[4] Let G be a connected graph on v vertices with $r+1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \ldots < \theta_r$. Let m be a nonnegative integer and let X and Y be sets of vertices, such that the distance between any vertex of X and any vertex of Y is at least $m+1$. Then*

$$\frac{|X||Y|}{(v-|X|)(v-|Y|)} \leq \min_{p \in P_m} \max_{i \neq 0} p^2(\theta_i).$$

First of all, this implies that the diameter of a graph is at most r, otherwise we would be able to find nonempty sets X and Y at distance $r+1$, and a polynomial $p \in P_r$ which is zero at the r nonzero eigenvalues, giving an upper bound zero, contradicting the theorem.

Using the theory of uniform approximations of continuous functions we were able to rewrite the upper bound as (cf. [3])

$$\min_{p \in P_m} \max_{i \neq 0} p^2(\theta_i) = \max_{T \subset \{1,\ldots,r\}, |T|=m+1} \left(\sum_{j \in T} \prod_{i \in T \setminus \{j\}} \frac{\theta_i}{|\theta_j - \theta_i|} \right)^{-2}.$$

3 Sets of vertices at given distance

A second application of the theorem now gives a bound on the number of vertices at distance r (hence at extremal distance) from an arbitrary vertex.

Theorem 2 *[3] Let G be a connected graph on v vertices with $r+1$ distinct Laplace eigenvalues $0 = \theta_0 < \theta_1 < \ldots < \theta_r$. Let x be an arbitrary vertex, and let k_r be the number of vertices at distance r from x. Then*

$$k_r \leq \frac{v}{1 + \frac{\gamma^2}{v-1}}, \text{ where } \gamma = \sum_{j \neq 0} \prod_{i \neq 0, j} \frac{\theta_i}{|\theta_j - \theta_i|}.$$

Of course we should note that computing the number of vertices at distance r can be done in polynomial time.

It is, however, not hard to show that deciding whether there exist two equally large sets of vertices of size κ with no edges in between (disconnected vertex sets) is an NP-complete problem (cf. [5, problem GT24]). From our tool we derive that

$$\kappa \leq \frac{1}{2} v (1 - \frac{\theta_1}{\theta_r}),$$

by using the polynomial $p(z) = 1 - \frac{2z}{\theta_1 + \theta_r}$. Haemers [6] used this method to derive a bound due to Helmberg, Mohar, Poljak and Rendl [8] on the bandwidth of a graph. Note that computing the bandwidth is also an NP-complete problem (cf. [5, problem GT40]).

A similar problem is to find two sets of vertices of size κ_r which are at (extremal) distance r. Here we find that

$$\kappa_r \leq \frac{v}{1+\gamma}, \text{ where } \gamma = \sum_{j \neq 0} \prod_{i \neq 0, j} \frac{\theta_i}{|\theta_j - \theta_i|}.$$

A related problem is the problem of finding two sets of vertices with no edges in between (disconnected vertex sets) such that the product of the sizes of these sets is maximized. This problem has an application in information theory and is studied by Haemers [7]. By using Theorem 1 he finds that

$$\max_{X,Y \text{disconnected}} \sqrt{|X||Y|} \leq \frac{1}{2} v \left(1 - \frac{\theta_1}{\theta_r}\right).$$

We should note that all bounds mentioned so far are attained by infinitely many graphs.

4 The diameter

To obtain a bound on the diameter $d(G)$ of a graph G we shall not solve the maximization problem in the upper bound of Theorem 1, but a relaxation of this problem. Instead of evaluating the polynomials at the discrete values $\theta_1, \ldots, \theta_r$, we evaluate them at the interval $[\theta_1, \theta_r]$, so that for the upper bound we find

$$\min_{p \in P_m} \max_{i \neq 0} p^2(\theta_i) \leq \min_{p \in P_m} \max_{z \in [\theta_1, \theta_r]} p^2(z).$$

The solution of the relaxation can be described in terms of Chebyshev polynomials (cf. [10]). The polynomial $C_m(z) = \frac{T_m(\frac{\theta_r + \theta_1 - 2z}{\theta_r - \theta_1})}{T_m(\frac{\theta_r + \theta_1}{\theta_r - \theta_1})}$ where $T_m(z) = \cosh(m \cosh^{-1}(z))$, solves the problem, thus giving the following diameter bound.

Theorem 3 *[4] Let G be a connected noncomplete graph on v vertices with smallest nonzero Laplace eigenvalue θ_1 and largest Laplace eigenvalue θ_r, then*

$$d(G) \leq \frac{\cosh^{-1}(v-1)}{\cosh^{-1}(\frac{\theta_r + \theta_1}{\theta_r - \theta_1})} + 1.$$

We shall apply this bound to Ramanujan graphs. These are regular graphs, say of degree k, for which $\max_{i: 0 < \theta_i < 2k} |k - \theta_i| \leq 2\sqrt{k-1}$ (cf. [9]). Now it follows from Theorem 3 that for a nonbipartite Ramanujan graph G on v vertices we have

$$d(G) < \frac{2 \log 2(v-1)}{\log(k-1)} + 1$$

(and for bipartite Ramanujan graphs we obtain a similar bound after applying an improved diameter bound for bipartite graphs (cf. [4])), which means that a Ramanujan graph has a small diameter, since the upper bound is approximately twice a (trivial) lower bound for the diameter of any k-regular graph on v vertices.

The diameter bound of Theorem 3 also has an interesting application in coding theory. Using the coset graph of a linear code, it gives a bound for the covering radius of the code in terms of its dual weights (cf. [4]).

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